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COMPLETENESS OF DERIVATIVES OF SQUARED SCHRÖDINGER EIGENFUNCTIONS AND EXPLICIT SOLUTIONS OF THE LINEARIZED KdV EQUATION

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April 1981

(Received January 29, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Explicit solutions to the Cauchy problem for the linearized KdV equation are constructed when the initial data is integrable. The method is analogous to the Fourier decomposition for a constant coefficient equation and uses the connection between the one-dimensional Schrödinger equation and the KdV equation, as discovered by Gardner, Greene, Kruskal, and Miura [2]. An expansion theorem expressing any integrable function in terms of derivatives of squared Schrödinger (generalized) eigenfunctions is proved. These functions evolve according to the linearized KdV equation, hence the expansion of the initial data leads to a generalized solution of the linearized KdV equation. Under suitable restrictions on the initial data, the solution constructed is classical. The proof of the expansion theorem may be interpreted as the skew-adjoint analogue of the more familiar process of simultaneously diagonalizing two self-adjoint operators.

AMS (MOS) Subject Classification: 35C15, 35Q20, 34B25

Key Words: Linearized KdV equation; Schrödinger eigenfunctions; skew-symmetric operators

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by an A.M.S. Postdoctoral Research Fellowship.

SIGNIFICANCE AND EXPLANATION

The Korteweg-deVries equation (KdV for short) arises as an approximation in many non-linear wave problems with weak dispersion and weak nonlinearity. We present a method for constructing explicit solutions of the linearized KdV equation. This equation is of importance in studying the effects of perturbations of the KdV equation. The method relies on the connection between the KdV equation and the one-dimensional Schrödinger equation, as discovered by Gardner, Greene, Kruskal, and Miura [2]. An expansion of the initial data in terms of derivatives of squared (generalized) eigenfunctions of the Schrödinger equation provides a decomposition of the solution resembling the use of Fourier transforms in solving constant coefficient equations. For the linearization about the zero solution of the KdV equation, the analogy holds in the strict sense, as our expansion reduces to the usual Fourier transform.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

COMPLETENESS OF DERIVATIVES OF SQUARED SCHRÖDINGER
EIGENFUNCTIONS AND EXPLICIT SOLUTIONS OF THE LINEARIZED KdV EQUATION

Robert L. Sachs

1. INTRODUCTION

In this paper, we present an explicit solution to the Cauchy problem for the linearized KdV equation:

$$(*)\quad \begin{cases} u_t + u_{xxx} - 6(qu)_x = 0 \\ \\ u(x,0) = \phi(x) \end{cases}$$

where $q(x,t)$ is a solution of the KdV equation ((1.5) below). Our method expresses the solution as a superposition of particular solutions and utilizes a completeness theorem which we discuss below. The particular solutions we choose may be thought of as derivatives of $q(x,t)$ with respect to the scattering data for the Schrödinger equation with potential $q(x,t)$. Hence we sketch briefly the inverse scattering method of solving the KdV equation, as discovered by Gardner, Greene, Kruskal, and Miura [2].

If we consider the one-dimensional Schrödinger equation with potential $Q(x)$,

$$(1.1)\qquad -\frac{d^2}{dx^2} f + Q(x) f = k^2 f$$

and define the Jost solutions $f_{\pm}(x,k)$ by their asymptotic behavior

$$(1.2)\qquad \begin{aligned} f_+ &\sim e^{ikx} \quad \text{as} \quad x \to +\infty \\ f_- &\sim e^{-ikx} \quad \text{as} \quad x \to -\infty \end{aligned}$$

then the relation

$$(1.3)\qquad T(k)\ f_-(x,k) = f_+(x,-k) + R(k)\ f_+(x,k)$$

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which defines the transmission coefficient $T(k)$ and the reflection coefficient $R(k)$ implies $T(k)$ is meromorphic in $\text{Im } k > 0$ with finitely many poles, all on the imaginary axis. The completeness theorem mentioned above expresses any integrable function ϕ in terms of $(f_+^2)'(x,k)$, $(f_-^2)'(x,k)$ and a sum of discrete terms related to the poles of $T(k)$. (While we could use (1.3) to eliminate $(f_-^2)'(x,k)$, it is more convenient not to do so.) We prove the theorem by solving the equation

$$\psi''' - 4Q\psi' - 2Q'\psi + 4k^2\psi' = \phi$$

for ψ' and integrating the 'resolvent'.

If we now consider a one-parameter family of Schrödinger operators

$$L(t) \equiv -\frac{d^2}{dx^2} + q(x,t) \tag{1.4}$$

where the time evolution of $q(x,t)$ is given by the KdV equation

$$q_t + q_{xxx} - 6qq_x = 0 \tag{1.2}$$

then it turns out [2] that for any t, $L(t)$ is unitarily equivalent to $L(0)$. This implies that the spectrum of $L(t)$ is invariant. Moreover, the scattering data associated with the operators $L(t)$ evolves in a very simple manner. Zakharov and Faddeev [15] interpret the above facts in the context of completely integrable Hamiltonian systems and show that the eigenvalues of L and k times the logarithm of $|T(k)|^2$ for k real form action variables with appropriate conjugate angle variables. In [15], a formal calculation appears which expresses the infinitesmal variation of $q(x,0)$ in terms of variations of the scattering data. This formula suggests consideration of x-derivatives of the squared eigenfunctions of (1.1) with their induced time dependence as solutions to the linearized KdV equation. The fact that these derivatives satisfy the linearized KdV equation for smooth potentials already appears implicitly in [2], Theorem 3.6. A related expansion for the Zakharov-Shabat eigenvalue problem appears in Kaup [4]. Discussions of perturbations using the inverse scattering formalism appear in [5], [9] (for the sine-Gordon equation), and [10]; for an application of this result to the problem of water waves in a canal, see [12]. Squared eigenfunctions and their derivatives also play an important role in the theory of the periodic KdV equation [8].

The completeness theorem is proved in Section 2 below, while in Section 3, the time evolution of the eigenfunctions and the solution of (*) are discussed. Some of these results appear in the author's doctoral dissertation (N.Y.U., October 1980). The advice and encouragement of his advisor, Jürgen Moser, is gratefully acknowledged.

We also remark that semi-group methods [3] will yield a solution of (*) for L^2 initial data, so for a large class of initial data, we have constructed the "evolution operator" explicitly.

2. L^1-COMPETENESS OF DERIVATIVES OF SQUARED SCHRÖDINGER EIGENFUNCTIONS

After introducing some notation and results from the scattering theory of the one-dimensional Schrödinger equation, we state and prove an expansion theorem for derivatives of squared Schrödinger eigenfunctions. (We shall use the term eigenfunction to include generalized eigenfunctions as well as *bona fide* L^2 solutions.)

Consider the Schrödinger equation

(2.1) $$-f''(x,k) + Q(x)\, f(x,k) = k^2 f(x,k)$$

for k real. Our notation shall be:

$$f'(x,k) = \frac{\partial}{\partial x} f(x,k)$$

$$\dot{f}(x,k) = \frac{\partial}{\partial k} f(x,k)$$

We assume the potential $Q(x)$ satisfies

(2.2) $$\|Q\|_{L^1_2} \equiv \int_{-\infty}^{\infty} (1 + x^2)\,|Q(x)|\, dx < \infty$$

The fundamental discovery of Gardner, Greene, Kruskal, and Miura [2], later formulated abstractly by Lax [7], is that if $q(x,t)$ evolves according to the KdV equation, the spectrum of the Schrödinger equation (2.1) with potential $q(x,t)$ remains fixed in t and the associated scattering data evolves in a very simple manner. We shall use this information below, but first introduce some notation and basic facts about scattering theory for (2.1). This information (and much more) may be found in [1].

Let $f_{\pm}(x,k)$ denote the Jost solutions of (2.1)--
i.e. $f_+(x,k) \sim e^{ikx}$ as $x \to +\infty$, $f_-(x,k) \sim e^{-ikx}$ as $x \to -\infty$, and both satisfy (2.1). The transmission coefficient, $T(k)$, as defined in (1.3) above, is represented in terms of the Wronskian of f_+, f_- by:

(2.3) $$\frac{1}{T(k)} = \frac{1}{2ik}\,[f_+(x,k),\, f_-(x,k)] = \frac{f_+'f_- - f_-'f_+}{2ik}$$

Formula (2.3) and the normalization of f_+, f_- imply that $T(k)$ is meromorphic in the upper half-plane $\text{Im } k > 0$ with poles at $k = i\beta_j$, $j = 1, \ldots, N$ where each energy $-\beta_j^2$ is a bound state energy in (2.1). N is finite by a classical estimate assuming $(1+|x|)|Q(x)|$ is integrable. $T(k)$ is also continuous and nonzero for real $k \neq 0$. For notational ease, we also introduce for $j = 1, \ldots, N$ the following pair of functions:

$$(2.4) \quad F_j(x) = f_+^2(x,i\beta_j) \ ; \quad G_j(x) = c_j f_+(x,i\beta_j)\cdot g_j(x)$$

where $g_j(x) \equiv \frac{1}{i}\frac{d}{dk}\left[f_-(x,k) - \frac{f_-(x,i\beta_j)}{f_+(x,i\beta_j)} f_+(x,k)\right]_{k=i\beta_j}$

and c_j is chosen so that $\int_{-\infty}^{\infty} F_j'(x)\, G_j(x)\, dx = 1$ for $j=1, \ldots, N$. The expansion theorem mentioned above is:

<u>Theorem 2.1</u> Suppose $Q(x)$ satisfies (2.2). If $\phi(x)$ is continuous and in L^1, then

$$(2.5)(a) \quad \phi(x) = \lim_{\varepsilon\downarrow 0} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{dk}{2\pi i k} T^2(k)\cdot \int_{-\infty}^{\infty} K(x,y,k)\, \phi(y)\, dy$$

$$+ \sum_{j=1}^{N} \int_{-\infty}^{\infty} [F_j'(x)\, G_j(y) - G_j'(x)\, F_j(y)]\, \phi(y)\, dy$$

and

$$(2.5)(b) \quad \phi(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{T^2(k)}{4\pi i k} [(f_+^2)'(x,k) f_-^2(y,k) - (f_-^2)'(x,k) f_+^2(y,k)]\, \phi(y)\, dy\, dk$$

$$+ \sum_{j=1}^{N} \int_{-\infty}^{\infty} [F_j'(x)\, G_j(y) - G_j'(x)\, F_j(y)]\, \phi(y)\, dy$$

where the kernel $K(x,y,k)$ is defined as:

$$
(2.6) \qquad K(x,y,k) \equiv \begin{cases} \frac{\partial}{\partial x} (f_+^2(x,k) f_-^2(y,k) - h(x,k)h(y,k)) & \text{for } y \le x \\ \\ \frac{\partial}{\partial x} (h(x,k)h(y,k) - f_-^2(x,k) f_+^2(y,k) & \text{for } y \ge x \end{cases}
$$

with $h(x,k) \equiv f_+(x,k) f_-(x,k)$.

We remark that for $q = 0$, $f_\pm(x,k) = e^{\pm ikx}$ and the above expansion reduces to the ordinary Fourier transform. The latter representation (2.5)(b) is more convienient in applications while (2.5)(a) is central to the proof of the theorem. Before presenting the proof, we discuss the choice of the particular kernel $K(x,y,k)$ of (2.6).

If f and g are both C^3 solutions of the Schrödinger equation (2.1) for the same energy k^2, then their product $f \cdot g$ is a solution of the third-order equation:

$$
(2.7) \qquad \psi''' - 4Q\psi' - 2Q'\psi = -4k^2\psi'
$$

Two linearly independent solutions of (2.1) generate three independent solutions of (2.7) e.g. f^2, fg, g^2. We choose $f_+(x,k)$, $T(k) \cdot f_-(x,k)$. Then, solving the inhomogeneous form of (2.7) for a function $\phi(x)$ by variation of parameters leads to an expression for ψ in terms of ϕ. The kernel $K(x,y,k)$ is the 'Green's function' for this problem. Differentiating, we obtain formally

$$
(2.8) \qquad \psi' = (D^2 - 4(Q - k^2) - 2Q'D^{-1})^{-1}\phi
$$

which we integrate as though it were a *bona fide* resolvent and obtain a multiple of the identity. The calculation of this integral is the content of the following:

<u>Lemma 2.2.</u> Let Γ_R be the semicircle in the upper half-plane of radius R traversed from $-R$ to R. Then:

$$\text{(2.9)} \qquad \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_R} \frac{T^2(k)}{k} \left\{ \int_{-\infty}^{\infty} K(x,y,k)\ \phi(y)\ dy \right\} dk$$

$= \phi(x)$ for all ϕ which are continuous with $\phi \in L^1$.

Postponing the proof of Lemma 2.2 momentarily, we show that Lemma 2.2 implies Theorem 2.1.

<u>Proof of Theorem 2.1, given Lemma 2.2</u>: Apply Cauchy's theorem. For $\operatorname{Im} k > 0$, the integrand in k has poles only at the poles of $T^2(k)$; an easy calculation shows that for $k = i\beta_1, \ldots, i\beta_N$ (recall, the bound state energies are $-\beta_N^2 < -\beta_{N-1}^2 < \ldots < -\beta_1^2 < 0$), the pole of $T(k)$ is simple [1]. Thus the integrand we consider has double poles at $k = i\beta_j$, $j = 1, \ldots, N$.

For $k = i\beta_j$, there is a constant α_j such that $f_-(x,i\beta_j) = \alpha_j f_+(x,i\beta_j)$. Thus the quantities:

$$\text{(2.10)} \qquad f_+^2(x,k)\ f_-^2(y,k) - h(x,k)\ h(y,k) \quad \text{and}$$

$$f_+(x,k)\ f_-(y,k) \quad - f_-(x,k) f_+(y,k)$$

vanish identically in x,y when $k = i\beta_j$.

Let A_j be the residue of $T(k)$ at $k = i\beta_j$. Then the residue of the left-hand side of (2.9) at $k = i\beta_j$ is precisely:

$$\text{(2.11)} \qquad \frac{1}{2\pi i} \frac{A_j^2}{i\beta_j} \frac{d}{dk} \left\{ \int_{-\infty}^{\infty} K(x,y,k)\ \phi(y)\ dy \right\}_{k=i\beta_j}$$

$$= \frac{1}{2\pi i} \frac{A_j^2}{\beta_j} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \Big[f_+^2(x,i\beta_j) \cdot f_+(y,i\beta_j) g_j(y)$$

$$- f_+(x,i\beta_j) g_j(x)\ f_+^2(y,i\beta_j) \Big]\ \phi(y)\ dy$$

where we recall $g_j(x) \equiv \frac{1}{i}\frac{d}{dk}[f_-(x,k)-\alpha_j f_+(x,k)]_{k=i\beta_j}$.

The right-hand side of (2.11) comes from replacing $f_-(x,i\beta_j)$ by $\alpha_j f_+(x,i\beta_j)$ after using the remark in (2.10). Now we deform the semicircle to the real k-axis. By our definitions above, the deformation contributes the terms $\sum_{j=1}^{N} \int_{-\infty}^{\infty} [F_j'(x)G_j(y) - G_j'(x)F_j(y)]\ \phi(y)\ dy$ which appear in the conclusion of Theorem 2.1, in addition to the integration along the real k-axis. We also remark that despite appearances, *neither real* k *integral has a singularity at* $k=0$. This follows from the fact that either (i) $f_+(x,0)$ and $f_-(x,0)$ are linearly dependent, which by the same remark as above in (2.10) implies $K(x,y,k)$ vanishes at least linearly in k as $k \to 0$ in $\text{Im}\, k \geq 0$, or (ii) $f_+(x,0)$ and $f_-(x,0)$ are linearly independent, so by (2.3), $T(k) = \alpha k + o(k)$ as $k \to 0$. In either case, there is no pole at $k = 0$. (See [1] for a further discussion of phenomena at $k = 0$ in scattering theory.) Thus we have proved (2.5)(a) assuming Lemma 2.2. To obtain (2.5)(b), we remark that the difference between the two k-integrals integrates to 0.

The difference between (2.5)(a) - (2.5)(b) is precisely:

$$\int_{-\infty}^{\infty} \frac{dk}{4\pi ik} \int_{-\infty}^{\infty} \hat{K}(x,y,k)\ \phi(y)\ dy$$

where $\hat{K}(x,y,k) = \text{sgn}(x-y)\frac{\partial}{\partial x}\left[T(k)f_+(x,k)f_-(y,k) - T(k)f_-(x,k)f_+(y,k)\right]^2$

Using (1.3) to eliminate $f_-(x,k)$, $f_-(y,k)$, we see that $\hat{K}(x,y,k)$ is an even function of k, so the integral vanishes. This proves (2.5)(b).

To complete the argument, we now prove Lemma 2.2. Consider

$$(2.12) \qquad I_R \equiv \frac{1}{2\pi i} \int_{\Gamma_R} \frac{T^2(k)}{k} \left\{ \int_{-\infty}^{\infty} K(x,y,k) \quad \phi(y)\ dy \right\} dk$$

Write $f_{\pm}(x,k) = m_{\pm}(x,k)\, e^{\pm ikx}$. We shall make use of the following estimates, taken from Deift-Trubowitz [1], which hold for all k in $\mathrm{Im}\, k \geq 0$:

$$(2.13)\qquad \text{(i)}\quad |m_{\pm}(x,k)-1| \leq \exp\{C_1/|k|\}\,\frac{C_2}{|k|}$$

$$\text{(ii)}\quad |m_{\pm}'(x,k)| \leq C_3/(1+|k|)$$

$$\text{(iii)}\quad T(k) = 1 + O\left(\frac{1}{|k|}\right) \quad \text{as} \quad |k| \to \infty.$$

In [1] it is also shown that $m_{\pm}-1$ are Hardy functions; in particular, they are analytic in $\mathrm{Im}\, k > 0$. Recalling the definition of $K(x,y,k)$ given by (2.6), we define:

$$(2.14)\quad \begin{cases} I_R^{(1)} \equiv \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_R} dk\, \dfrac{T^2(k)}{k} \left\{ \displaystyle\int_{-\infty}^{x} 2ik\, m_+^2(x,k) m_-^2(y,k) \cdot e^{2ik(x-y)}\, \phi(y)\, dy \right. \\ \qquad \left. + \displaystyle\int_{x}^{\infty} 2ik m_-^2(x,k) m_+^2(y,k) e^{-2ik(x-y)}\, \phi(y)\, dy \right\} \\ I_R^{(2)} \equiv \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_R} dk\, \dfrac{T^2(k)}{k} \left\{ \displaystyle\int_{-\infty}^{x} [2m_+(k,x) m_+'(x,k) m_-^2(y,k) e^{2ik(x-y)} \right. \\ \qquad - h'(x,k)h(y,k)]\, \phi(y)\, dy \\ \qquad \left. + \displaystyle\int_{x}^{\infty} [h'(x,k)h(y,k) - 2m_-(x,k) m_-'(x,k) m_+^2(y,k) \cdot e^{-2ik(x-y)}]\, \phi(y)\, dy \right\} \end{cases}$$

Thus $I_R = I_R^{(1)} + I_R^{(2)}$.

By estimates (i), (ii), (iii) in (2.13), since $I_R^{(2)}$ contains terms which have a factor $m_{\pm}'(x,k)$ and $m_{\pm}(x,k)$ is uniformly bounded for $|k| > c > 0$, we have the estimate

(2.15) $\quad |I_R^{(2)}| \le C \frac{\|\phi\|_{L^1}}{R}$ for all $R \ge R_0$ sufficiently large, where C is independent of R.

Moreover, by (2.13) (i), (iii), $m_{\pm}(x,k) = 1 + O(\frac{1}{R})$. $T(k) = 1 + O(\frac{1}{R})$ for $|k| = R$ so

$$I_R^{(1)} = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{dk}{k} \left\{ \int_{-\infty}^{x} 2ik\, e^{2ik(x-y)}\, \phi(y)\, dy + \int_{x}^{\infty} 2ik\, e^{-2ik(x-y)}\, \phi(y)\, dy \right\} + O(\frac{1}{R})$$

The first terms converge to $\phi(x)$ as in the usual proof of Fourier completeness [14] and the lemma is proved by taking $R \to \infty$.

Remarks: (i) The expansion theorem above bears a strong resemblance to that of the Fourier transform for L^1 functions. However, since the underlying process is the "simultaneous diagonalization" of the two skew operators $\frac{d}{dx}$ and $-(\frac{d}{dx})^3 + 2\frac{d}{dx}Q + 2Q\frac{d}{dx}$, the analoque of the Fourier L^2 theory does not exist if $Q \neq 0$. If we define

(2.16) $$\hat{\phi}_{\pm}(k) = \int_{-\infty}^{\infty} \phi(y) f_{\pm}^2(y,k)\, dy \,,$$

the natural version of the Plancheral formula in this case relates the skew bilinear form $\int_{-\infty}^{\infty} \psi'(x)\ \phi(x)$ to the standard symplectic pairing

$$\left(\begin{pmatrix} \hat{\psi}_+ & (k) \\ \hat{\psi}_- & (k) \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi}_+ & (k) \\ \hat{\phi}_- & (k) \end{pmatrix} \right)$$

(ii) We also note that the skew operator $-D^3 + 2DQ + 2QD$ $\quad (D = \frac{d}{dx})$ was used by Lenard to recursively generate the KdV conservation laws [2]. It seems to play a crucial role in many aspects of the KdV theory -- e.g. it is useful in proving that the integrals are in involution.

3. APPLICATION OF THEOREM 2.1 TO THE CAUCHY PROBLEM FOR THE LINEARIZED KdV EQUATION

The expansion of $\phi(x)$ given by Theorem 2.1 will lead directly to a method for solving the Cauchy problem for the linearized KdV equation:

$$(*) \qquad u_t + u_{xxx} - 6(qu)_x = 0 ; \quad u(x,0) = \phi(x)$$

with ϕ satisfying the hypotheses of Theorem 2.1. As the potential $q(x,t)$ in the Schrödinger equation

$$- f'' + qf = k^2 f$$

evolves according to the KdV equation, the corresponding eigenfunctions evolve in time. Gardner, Greene, Kruskal, and Miura [2] observed that the squares of the eigenfunctions satisfy the formal adjoint of the linearized KdV equation (which they called "the associated linear equation"), namely:

$$(3.1) \qquad v_t + v_{xxx} - 6qv_x = 0$$

from which it follows that $u \equiv v_x$ satisfies

$$(3.2) \qquad u_t + u_{xxx} - 6(qu)_x = 0 \text{ , the linearized KdV equation.}$$

In view of this fact, the expansion of Theorem 2.1 may be extended to include the time evolution of the eigenfunctions. As we shall see below, this extension is the solution of the Cauchy problem (*).

We begin by developing some necessary preliminary facts:

<u>Lemma 3.1</u> The functions $g_j(x) \equiv \frac{1}{i}\frac{d}{dk}[f_-(x,k)-\alpha_j f_+(x,k)]$, $k = i\beta_j$ are (unbounded) solutions of the Schrödinger equation (2.1) with $k^2 = -\beta_j^2$.

<u>Proof</u>: Differentiating (2.1) with respect to k, we obtain an equation for $(d/dk)f_{\pm}(x,k) = \dot{f}_{\pm}(x,k)$:

(3.3) $$\dot{f}''_{\pm} = (q - k^2)\dot{f}_{\pm} - 2kf_{\pm}$$

Consider $g_j \equiv \frac{1}{i}[\dot{f}_- - \alpha_j \dot{f}_+]$ at $k = i\beta_j$: $g_j(x)$ satisfies

$$g_j'' = (q + \beta_j^2)g_j - 2\beta_j(f_-(x,i\beta_j) - \alpha_j f_+(x,i\beta_j))$$

$$= (q + \beta_j^2)g_j \text{ by our choice of } \alpha_j.$$

Remark: $g_j(x)$ is exponentially increasing as $|x| \to \infty$; however, the product $f_+(x,i\beta_j)\ g_j(x)$ is bounded.

We now discuss the result in [2] mentioned above, and sketch the proof.

Lemma 3.2 (cf. equation (2.19) of [2]).

Let ψ be a solution of the Schrödinger equation (2.1) with potential $q(x,t)$ evolving according to the KdV equation.

Then the function

(3.4) $$R \equiv \psi_t + \psi_{xxx} - 3(q + k^2)\psi_x$$

is also a solution of the Schrödinger equation with potential $q(x,t)$.

Sketch of the proof: Use the equation $\psi'' = (q-k^2)\psi$ to express q in terms of ψ. Substitute into the KdV equation and simplify the resulting expression, obtaining the equation $R'' - (q-k^2)R = 0$ after eliminating a factor of ψ.

Remark: The chief use of Lemma 3.2 is to show that R, for suitable eigenfunctions ψ, is in fact 0. In particular, we have:

Corollary 3.3 The expression R vanishes if we choose any of the following eigenfunctions for ψ:

(i) $\tilde{f}_{\pm}(x,k,t) \sim \exp\left\{\pm i(kx+4k^3 t)\right\}$ as $x \to \pm\infty$, t fixed;

(ii) $\tilde{f}_{+}(x,i\beta_j,t)$;

(iii) $g_j(x,t) \equiv \frac{1}{i}\frac{d}{dk}[\tilde{f}_-(x,k,t) - \alpha_j \tilde{f}_+(x,k,t)]_{k=i\beta_j}$

Sketch of proof (see [2], Theorem 3.6): Consider R in each case. By our choice of asymptotics, we have $R \equiv 0$ since no other solution of the Schrödinger equation with that type of decay exists, namely $R \to 0$ as $x \to \pm\infty$ in (i), $\exp\{|\beta_j x - 4\beta_j^3 t|\}R \to 0$ in (ii), (iii) as $|\beta_j x - 4\beta_j^3 t| \to \infty$. Thus from the spectral theory of the Schrödinger equation, $R \equiv 0$ in each case.

Lemma 3.4 Suppose ψ_1, ψ_2 are two (not necessarily independent) solutions of the Schrödinger equation for the same eigenvalue. If $\psi_t + \psi_{xxx} - 3(q + k^2)\psi_x \equiv 0$ for $\psi = \psi_j$, $j = 1,2$, then the product $\psi_1\psi_2$ is a solution of the adjoint equation:

$$v_t + v_{xxx} - 6qv_x = 0$$

Proof: A direct calculation:

$$
\begin{aligned}
&(\psi_1\psi_2)_t + (\psi_1\psi_2)_{xxx} - 6q(\psi_1\psi_2)_x \\
&= \psi_1(\psi_{2_t} + \psi_{2_{xxx}} - 6q\psi_{2_x}) + \psi_2(\psi_{1_t} + \psi_{1_{xxx}} - 6q\psi_{1_x}) \\
&\quad + 3\psi_{1_x}\psi_{2_{xx}} + 3\psi_{1_{xx}}\psi_{2_x} \\
&= \psi_1(\psi_{2_t} + \psi_{1_{xxx}} - 3(q+k^2)\psi_{2_x}) \\
&\quad + \psi_2(\psi_{1_t} + \psi_{1_{xxx}} - 3(q+k^2)\psi_{1_x})
\end{aligned}
$$

$= 0$, where we used the Schrödinger equation to eliminate the second derivatives.

Remarks: (i) An alternate derivation of these facts may be given using the following idea of Tanaka [13]: The KdV equation may be written in the Lax [7] form

$$\frac{dL}{dt} = [B,L] \tag{3.5}$$

where $L(t)$ is the operator $-\frac{d^2}{dx^2} + q(x,t)$ and $B(t)$ is the skew operator

$$-4 \frac{d^3}{dx^3} + 3q(x,t) \frac{d}{dx} + 3 \frac{d}{dx} q(x,t)$$

The time derivative of the Schrödinger equation (2.1) and (3.5) implies

(3.6) $$L(f_t - Bf) = k^2(f_t - Bf).$$

Choosing $f = f_+(x,k,t) \sim e^{ikx}$ as $x \to +\infty$ for t fixed and analyzing the asymptotic behavior of $f_t - Bf$ as $x \to +\infty$ for t fixed implies

(3.7) $$(f_+)_t - B(f_+) = 4(ik)^3(f_+)$$

so $\psi(x,k,t) \equiv e^{4ik^3t} \cdot f_+(x,k,t)$

satisfies

(3.8) $$\psi_t - B\psi = 0 \ , \quad \psi \sim e^{ikx + 4ik^3t} \quad \text{as } x \to +\infty \text{ for } t \text{ fixed}$$

(a similar argument holds for $e^{-4ik^3t} \cdot f_-(x,k,t)$). Using the Schrödinger equation (2.1), a simple calculation as in Lemma 3.4 above shows that products (with the same values of k^2) of solutions of (3.8) satisfy the adjoint equation

$$v_t + v_{xxx} - 6qv_x = 0.$$

(ii) If $q(x,t)$ is a classical solution of the KdV equation, the formal calculations above are sensible-- i.e. the eigenfunctions possess the necessary derivatives. This follows from the inhomogeneous form of the Schrödinger equation which these derivatives satisfy.

Using these facts, we make the following definitions (extending (2.4) to include time dependence):

$$
(3.9)\qquad \begin{cases}
\tilde{f}_+(x,k,t) \sim e^{ikx+4ik^3t} \quad \text{as} \quad x \to +\infty, \quad t \quad \text{fixed} \\
\tilde{f}_-(x,k,t) \sim e^{-ikx-4ik^3t} \quad \text{as} \quad x \to -\infty, \quad t \quad \text{fixed} \\
\tilde{f}_\pm \quad \text{satisfy (2.1) with potential} \quad q(x,t) \\
\tilde{F}_j(x,t) = f_+^2(x,i\beta_j,t) \\
\tilde{G}_j(x,t) = c_j \, \tilde{f}_+(x,i\beta_j,t) \cdot \tilde{g}_j(x,t)
\end{cases}
$$

The obvious candidate for the solution of (*) is the function $u(x,t)$ defined as follows:

$$
(3.10)\qquad u(x,t) \equiv \int_{-\infty}^{\infty} \frac{dkT^2(k)}{4\pi ik} [(\tilde{f}_+^2)'(x,k,t)\,\hat{\phi}_-(k) - (\tilde{f}_-^2)'(x,k,t)\,\hat{\phi}_+(k)]
$$

$$
+ \sum_{j=1}^{N} \int_{-\infty}^{\infty} [F_j'(x,t)G_j(y,0) - G_j'(x,t)F_j(y,0)]\,\phi(y)\,dy
$$

where $\hat{\phi}_\pm(k)$ are defined in (2.16) above. By Lemma 3.4, all of the functions of (x,t) appearing on the right-hand side of (3.10) satisfy the linearized KdV equation. Thus we have proved:

Lemma 3.5 The function $u(x,t)$ defined by (3.10) above satisfies the linearized KdV equation:

$$
(*)\qquad \begin{cases}
u_t + u_{xxx} - 6(qu)_x = 0 \\
u(x,o) = \phi(x)
\end{cases}
$$

in the sense of distributions.

To prove that $u(x,t)$ is a classical solution of (*) for $t > 0$, we need some additional smoothness and decay on $\phi(x)$, the initial data. The situation is completely analogous with the Fourier transform solution of the Airy equation:

$$w_t + w_{xxx} = 0 \tag{3.11}$$

The x-decay of ϕ is needed to have $\hat{\phi}_{\pm}(k)$ be differentiable while smoothness of ϕ relates to integrability of $k^{\alpha}\hat{\phi}_{\pm}(k)$ for $0 \le \alpha \le 2$. As in the case of (3.11), one can prove the following:

<u>Theorem 3.6</u> The function $u(x,t)$, given by (3.10) above, is a classical solution of the linearized KdV equation for $t > 0$ if

(i) $\phi(x)$ has four continuous derivatives.

(ii) As $|x| \to \infty$, $\partial_x^r \phi(x) = O(|x|^{-4})$ for $r = 0,1,2,3,4$.

<u>Sketch of proof</u>: Using (i) and the definition of $\hat{\phi}_{\pm}(k)$, we integrate by parts four times with respect to x, which implies, as $|k| \to \infty$, $\hat{\phi}_{\pm}(k) = O(|k|^{-4})$. Also, by (ii), $\hat{\phi}_{\pm}(k)$ are C^2. Thus $u(x,t)$ has two continuous derivatives with respect to x. As in Murray [11], the factor k^2 is written as $\dfrac{x + 12k^2t}{12t} - \dfrac{x}{12t}$, where the first term is a multiple of the k-derivative of the exponentials $e^{\pm 2ik(x+4k^2t)}$. Integrating by parts in k, we find that $u(x,t)$ has four continuous x-derivatives. Repeating the argument, we obtain six x-derivatives. From the linearized KdV equation, this implies u_t is continuous, hence it is a classical solution. The only difference between this case and the Airy equation (3.11) is the presence of the added factors

$$\tilde{m}_{\pm}(x,k,t) = \tilde{f}_{\pm}(x,k,t)\, e^{\mp 2ik(x+4k^2t)}$$

and these do not affect the necessary estimates.

A fuller discussion and an analysis of asymptotic behavior is presented for the N-soliton linearization in [12], where the perturbation theory for the problem of water waves in a canal is discussed. The KdV equation was first derived to model precisely this situation [6].

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TSR # 2207	2. GOVT ACCESSION NO.: AD-A100 610
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Completeness of Derivatives of Squared Schrödinger Eigenfunctions and Explicit Solutions of the Linearized KdV Equation	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Robert L. Sachs	8. CONTRACT OR GRANT NUMBER(s): DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 1(Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709	12. REPORT DATE: April 1981
	13. NUMBER OF PAGES: 18
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linearized KdV equation; Schrödinger eigenfunctions; skew-symmetric operators

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Explicit solutions to the Cauchy problem for the linearized KdV equation are constructed when the initial data is integrable. The method is analogous to the Fourier decomposition for a constant coefficient equation and uses the connection between the one-dimensional Schrödinger equation and the KdV equation, as discovered by Gardner, Greene, Kruskal, and Miura [2]. An expansion theorem expressing any integrable function in terms of derivatives of squared Schrödinger (generalized) eigenfunctions is proved. These functions

20. Abstract, continued

evolve according to the linearized KdV equation, hence the expansion of the initial data leads to a generalized solution of the linearized KdV equation. Under suitable restrictions on the initial data, the solution constructed is classical. The proof of the expansion theorem may be interpreted as the skew-adjoint analogue of the more familiar process of simultaneously diagonalizing two self-adjoint operators.

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